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On Totally Umbilical Pseudo-Slant Submanifolds in Bronze Riemannian Manifold

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Abstract
This paper explores the geometry of totally umbilical pseudo-slant submanifolds in Bronze Riemannian manifolds. Existence conditions and geometric properties are presented, supported by examples.

Keywords: *Bronze manifold, pseudo-slant submanifold.*

1. Introduction

An $f-$ structure on a manifold refers to a $(1,1)-$ tensor field of constant rank, originally introduced by Yano in [21]. It satisfies the equation $f^3+f=0$, and this concept generalizes both almost contact and almost complex structures. Later, Goldberg and yano extended this concept by examining a polynomial structure of degree l for a $(1,1)$ tensor field f of constant rank on $\widetilde{M}$ satisfied the equation: [10].

$$\Theta(f)=f^l+a_l f^{l-1}+a_2 f+a_1 I=0.$$

Here I is identity tensor of $(1,1)-$type, and $a_l,\ldots,a_2,a_1$ are real numbers.

Inspired by the intriguing characteristics of the golden ratio, $\varphi=\frac{1+\sqrt{5}}{2}=1.618$ which is a positive root of the quadratic equation $x^2-x-1=0$, a novel concept called the golden structure on manifolds was introduced and examined by Hretcanu [11].

This approach involves the development of an associated almost product structure. Subsequently, Crasmareanu and collaborators [9] delved into the field of golden differential geometry, uncovering several significant findings. Inspired by the silver ratio $\theta=1+\sqrt{2}=2.414$ a positive root of $x^2-2x-1=0$. In 2016, Ozkan et al [15] introduced a new structure on manifolds known as the silver structure.

Building on the ideas of golden and silver structures on manifolds, we have recently explored the concept of a Bronze structure on manifolds [17] inspired by the Bronze ratio

$$\phi=\frac{3+\sqrt{13}}{2}=3.302,$$

which is the positive root of the equation

$$x^2-3x-1=0.$$

Notably, the golden ratio is recognized for its exceptionally slow convergence, making it the most "irrational" of all irrational numbers [20]. This unique property enhances the appeal of studying the silver and Bronze ratios, as their faster convergence characteristics introduce intriguing mathematical properties and make their exploration particularly compelling [4, 5].

In this paper, we assume that all manifolds, connections, and tensor fields are differentiable and belong to the specified class.

In the late 20th century, Chen [7, 8] developed the concept of slant submanifolds within the framework of almost Hermitian manifolds, as detailed in references [18]. A. Lotta later expanded this idea to contact metric manifolds [14], and Cabrerizo et al. extended it further to include slant submanifolds of K-contact and Sasakian manifolds [6]. Moreover, semi-slant and slant submanifolds of metallic Riemannian manifolds were examined in [12].

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The notion of semi-slant submanifolds in almost Hermitian manifolds was first introduced by Papaghuic in 2009 [19]. Likewise, hemi-slant submanifolds were initially defined by A. Carrizo and are also referred to as pseudo-slant submanifolds. In more recent years, Dirik and his collaborators have explored pseudo-slant submanifolds in various types of manifolds, as documented in their works from 2014 and 2016 [1, 2].

This study focuses on pseudo-slant submanifolds within the framework of Bronze Riemannian manifolds. Section 2 presents the essential definitions and foundational concepts. Section 3 delves into significant results concerning submanifolds in Riemannian manifolds equipped with a Bronze structure. In Section 4, a comprehensive characterization of totally umbilical pseudo-slant submanifolds in Bronze Riemannian manifolds is provided. The paper concludes by offering illustrative examples of non-trivial pseudo-slant submanifolds in such manifolds.

2. Preliminers

In this section, we present specific definitions and notations pertaining to Bronze Riemannian manifolds.

Definition 1. Let $\widetilde{\mathrm{M}}$ be a C^{∞} -manifold. If a tensor field φ of type (1,1) satisfies the equation

$$\varphi^2 = \varphi + I$$

then φ is called a golden structure on $\widetilde{\mathrm{M}}$ and , $(\widetilde{\mathrm{M}}, \varphi)$ is the golden manifold [11].

Definition 2. Let $\widetilde{\mathrm{M}}$ be a C^{∞} -manifold. A (1,1)-tensor field θ that satisfies the equation

$$\theta^2 = 2\theta + I$$

is referred to as a Bronze structure on $\widetilde{\mathrm{M}}$ and $(\widetilde{\mathrm{M}}, \theta)$ is the silver manifold [16].

Definition 3. Let $\widetilde{\mathrm{M}}$ be a C^{∞} -manifold. A (1,1)-tensor field ϕ that satisfies the equation

$$\phi^2 = 3\phi + I \tag{2.1}$$

is referred to as a Bronze structure on $\widetilde{\mathrm{M}}$ and $(\widetilde{\mathrm{M}}, \phi)$ is the Bronze manifold. Where I denotes the identity map [17].

Proposition 1.

- ϕ and $3-\phi$ are the eigenvalues of the Bronze structure .
- The Bronze structure ϕ is an isomorphism on $T_p\widetilde{\mathrm{M}}, \forall p \in \widetilde{\mathrm{M}}$.
- Consequently, ϕ is invertible and its inverse $\phi^{-1} = \widehat{\Psi}$ verifies the following:

$$\widehat{\Psi}^2 = -3\widehat{\Psi} + I$$

We now present the following theorem, which demonstrates a connection between the Bronze structure and the almost product structure of the manifold $\widetilde{\mathrm{M}}$ [17].

If ϕ represents a Bronze structure on a manifold $\widetilde{M}$, then the expression.

Theorem 1. Let P represent an almost product structure. In this case, P defines a Bronze structure on the manifold as follows.

$$\phi = \frac{1}{2}(3I + \sqrt{13}P)$$

moreover, if ϕ denotes a Bronze structure on , then

$$P = \frac{1}{\sqrt{13}}(2\phi - 3I)$$

gives an almost product structure on manifold $\widetilde{\mathrm{M}}$ [18].

Consider P as an almost product structure on a manifold $\widetilde{M}$, and g as a Riemannian metric satisfies:

$$g(PX, PY) = g(X, Y) \tag{2.2}$$

for any $X, Y \in \Gamma(T\widetilde{M})$.
Alternatively, P can be considered as a g-symmetric tensor, defined as:

$$g(PX, Y) = g(X, PY) \tag{2.3}$$

for any $X, Y \in \Gamma(T\widetilde{M})$. Here, (g, P) is called a Riemannian almost product structure.

ϕ is referred to as the Bronze structure. If the Riemannian metric g is ϕ harmonious, then $(\widetilde{M}, g, \phi)$ is called a Bronze Riemannian manifold [18]. For $\phi-$ harmonious metric, we get

$$g(\phi X, Y) = g(X, \phi Y) \tag{2.4}$$

for any $X, Y \in \Gamma(T\widetilde{M})$. If the interchange X and ϕX in (2.4), then , we have

$$g(\phi X, \phi Y) = g(\phi^2 X, Y) = g(3\phi X + X, Y) = 3g(\phi X, Y) + g(X, Y) \tag{2.5}$$

Example 1. Let $\mathbb{R}^4$ denote the Euclidean 4-space with standard coordinates (u_1, u_2, u_3, u_4). Consider ϕ a (1,1)-tensor field defined on $\mathbb{R}^4$.

$$\phi(u_1, \mathrm{u}, u_3, u_4) = (\phi u_1, \phi u_2, (3-\phi)u_3, (3-\phi)u_4)$$

for any vector field $(u_1, u_2, u_3, u_4) \in \mathbb{R}^4$, where $\phi = \frac{3+\sqrt{13}}{2}$ and $3-\phi = \frac{3-\sqrt{13}}{2}$ are the roots of $x^2 - 3x -$

$1 = 0$. To understand the structure of this tensor, we can look at its matrix representation. The tensor field ϕ maps the vector field as follows, corresponding to the matrix D:

$$D=\begin{pmatrix} \phi & 0 & 0 & 0 \\ 0 & \phi & 0 & 0 \\ 0 & 0 & 3-\phi & 0 \\ 0 & 0 & 0 & 3-\phi \end{pmatrix}$$

The eigenvalues of this matrix are ϕ and $3-\phi$. Then we obtain Thus, we have $\phi^2 - 3\phi - I = 0$. Moreover, we get

$$\begin{aligned}
&\langle \phi(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), (t_1, t_2, t_3, t_4) \\
&\quad = \langle ((\phi u_1, \phi u_2, (3-\phi)u_3, (3-\phi)u_4), (t_1, t_2, t_3, t_4)\rangle \\
&= \phi u_1 t_1 + \phi u_2 t_2 + (3-\phi)u_3 t_3 + (3-\phi)u_4 t_4 \\
&= \phi t_1 u_1 + \phi t_2 u_2 + (3-\phi)t_3 u_3 + (2-\phi)t_4 u_4 \\
&= \langle (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), (\phi t_1, \phi t_2, (3-\phi)t_3, (3-\phi)t_4)\rangle \\
&= \langle (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), \phi(t_1, t_2, t_3, t_4)\rangle
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle \phi X, \phi Y\rangle &= \langle \phi^2 X, Y\rangle = \langle 3\phi X + X, Y\rangle \\
&= \langle 3\phi(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4) + (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), (t_1, t_2, t_3, t_4)\rangle \\
&= \langle \big(3(\phi u_1, \phi u_2, (3-\phi)u_3, (3-\phi)u_4) + (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4)\big), (t_1, t_2, t_3, t_4)\rangle,
\end{aligned}$$

$$\begin{aligned}
&= \langle (3(\phi u_1, \phi u_2, (3-\phi)u_3, (3-\phi)u_4), (t_1, t_2, t_3, t_4)))\rangle + \langle (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), (t_1, t_2, t_3, t_4)\rangle \\
&= \langle 3\phi(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), (t_1, t_2, t_3, t_4)\rangle + \langle (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), (t_1, t_2, t_3, t_4)\rangle = \langle 3\phi X, Y\rangle + \langle X, Y\rangle.
\end{aligned}$$

for each vector fields $(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), (t_1, t_2, t_3, t_4) \in \mathbb{R}^4$. Hence, $(\mathbb{R}^4, \langle,\rangle, \phi)$ is a Bronze Riemannian manifold [3].

Theorem 2. Let $(\widetilde{\mathrm{M}}, g, \phi)$ represent a Bronze Riemannian manifold. The Bronze structure ϕ is said to be integrable if and only if $\widetilde{\nabla}\phi = 0$.

3. Submanifolds of a Bronze Riemannian Manifold

Submanifolds of a Bronze Riemannian manifold are entities that retain the manifold's geometric and metric properties, defined by a unique tensor structure associated with the Bronze ratio.

Let M be a submanifold of a Beonze Riemannian manifold $(\widetilde{M}, g, \phi)$, here g metric on M. Furthermore, let ∇ and $\nabla^{\perp}$ be the connections on TM and $T^{\perp}M$ of M, respectively. In this context, the Gauss and Weingarteen formulas can be stated as follows:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{3.1}$$

$$\widetilde{\nabla}_X V = -A_V X + \nabla_X^{\perp} V \tag{3.2}$$

for all $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$.
h and A_V are connected by the following relationship.

$$g(A_V X, Y) = g(h(X, Y), V) \tag{3.3}$$

for all $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$. The mean curvature vector H of M is given by

$$K = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i) \tag{3,4}$$

Here m =dim(M), $sp\{e_1, e_2, \ldots, e_m\}$ is a local orthonormal frame of M.

Let (M, g) be a submanifold of a Bronze Riemannian manifold $(\widetilde{M}, g, \phi)$. The submanifold M is said to be totally umbilical if h satisfies

$$h(X, Y) = g(X, Y)K \tag{3.5}$$

for all $X, Y \in \Gamma(TM)$, here K is the mean curvature vector. A submanifold M is said to be totally geodesic if the second fundamental form $h = 0$, and the manifold M is said to be minimal if $K = 0$.

Let (M, g) be a submanifold of a Bronze Riemannian manifold $(\widetilde{M}, g, \phi)$. Then, we get

$$\phi X = TX + NX, \tag{3.6}$$

In this context, TX represents the tangential part, while NX denotes the normal part of ϕX, for all $X \in \Gamma(TM)$. Similary , we get

$$\phi V = tV + nV \tag{3.7}$$

In this context, tV represents the tangential part, while nV denotes the normal part of ϕ, for all $V \in \Gamma(T^{\perp}M)$.

Proposition 2. Let M be a submanifold of Bronze Riemannian manifold $(\widetilde{\mathrm{M}}, g, \phi)$. Then, we get

$$g(TX, Y) = g(X, TY) \tag{3.8}$$
$$g(nU, V) = g(U, nV) \tag{3.9}$$
$$g(NX, V) = g(X, tV)$$

for any $X, Y \in \Gamma(TM)$ and for U, $V \in \Gamma(TM)^{\perp}$.

From (2.5) , we easily see that

$$g(TX,TY)+g(NX,NY)=g(X,Y)+3g(TX,Y). \quad (3.10)$$

Thus by using (2.1), (3.6) and (3.7), we obtain

$$T^2X=3TX+X-tNX,\ 3NX= NTX+nNX \quad (3.11)$$

and

$$3tV=TtV+tnV,\ n^2V=3nV+V-NtV \quad (3.12)$$

If M is $\phi-$ invariant, thus $N=0$. Thus from (3.11) and (3.12), we obtain

$$T^2=3T+I,\ n^2=3n+I \quad (3.13)$$

Therefore, (T,g) and (n,g) forms a Bronze structure on M.

Here, the covariant derivatives of T,N,t and n are defined as follows:

$$(\nabla_XT)Y=\nabla_XTY-T\nabla_XY \quad (3.14)$$

$$(\nabla_XN)Y=\nabla_X^{\perp}NY-N\nabla_XY \quad (3.15)$$

$$(\nabla_Xt)V=\nabla_XtV-t\nabla_X^{\perp}V \quad (3.16)$$

and

$$(\nabla_Xn)V=\nabla_X^{\perp}nV-n\nabla_X^{\perp}V \quad (3.17)$$

For any $X,Y\in\Gamma(TM)$.

Through direct calculations, the following formulas are obtained:

$$(\nabla_XT)Y=A_{NY}X+th(X,Y) \quad (3.18)$$

and

$$(\nabla_XN)Y=nh(X,Y)-h(X,TY) \quad (3.19)$$

Similary, for all $V\in\Gamma(T^{\perp}M)$, we have

$$(\nabla_Xt)V=A_{nV}X-TA_VX \quad (3.20)$$

and

$$(\nabla_Xn)V=-h(tV,X)-NA_VX \quad (3.21)$$

Corollary 1. Let M be a totally umbilical submanifold of a Bronze Riemannian manifold $(\widetilde{M},g,\phi)$. If M are $\phi-$anti invariant and invariant submanifold, the following properties are satisfied:

If M is $\phi-$ invariant submanifold	If M is $\phi-$anti$-$ invariant submanifold
$N=0$	$T=0,$
$(\nabla_XT)Y=tg(X,Y)K,$	$(\nabla_XN)Y=ng(X,Y)K,$
$(\nabla_Xn)V=-g(tV,X)K$	$(\nabla_Xt)V=g(K,nV)X$
$ng(X,Y)K=g(X,TY)K$	$g(K,NY)X=-g(X,Y)tK$
$g(K,nV)Y=g(K,V)TY$	$g(K,NY)Z=-g(K,NZ)Y$

for all $X,Y,Z\in\Gamma(TM)$, for all $V\in\Gamma(T^{\perp}M)$.

4. Totally Umbilical Pseudo-Slant Submanifolds of a Bronze Riemannian Manifold

Certain properties of totally umbilical pseudo-slant submanifolds within a Bronze Riemannian manifold have been described.

Definition 2. Let (M,g) be a submanifold of a Bronze Riemannian manifold $(\widetilde{M},g,\phi)$. For each $X\neq 0$ tangential to M at x, the angle $\beta(x)\in\left[0,\frac{\pi}{2}\right]$, between ϕX and T_xM is called the slant angle of M. If this slant is constant, the submanifold is known as a slant submanifold. When $\beta=0$ the submanifold is called an invariant submanifold, and when $\beta=\frac{\pi}{2}$, it is called an anti-invariant submanifold. If the slant angle $\beta(x)\in(0,\frac{\pi}{2})$ then the submanifold is classified as a proper-slantsubmanifold [6].

Theorem 3. Let (M,g) be a submanifold of a Bronze Riemannian manifold $(\widetilde{M},g,\phi)$. M is considered a slant submanifold if and only if there exists a constant $\lambda\in[0,1]$ such that:

$$T^2=\lambda(3\phi+I) \quad (4.1)$$

and

$$\phi^2=\frac{1}{\lambda}T^2 \quad (4.2)$$

furthermore, if β slant angle of M , then $\lambda=cos^2\beta$ [6].

Lemma 1. Let (M,g) be a submanifold of a Bronze Riemannian manifold $(\widetilde{M},g,\phi)$. Then, we have

$$g(TX,TY)=cos^2\beta\{g(X,Y)+3g(X,TY)\} \quad (4.3)$$

And

$$g(NX,NY)=sin^2\beta\{g(X,Y)+3g(TX,Y)\} \quad (4.4)$$

for all $X,Y\in\Gamma(TM)$ [6].

Definition 3 . Let M be a submanifold of a Bronze Riemannian manifold $(\widetilde{M},g,\phi)$. M is pseudo-slant submanifold if there exist two orthogonal distributions D_β and $D^\perp$, exist on M such that

- The tangent bundle TM has an orthogonal direct sum decomposition expressed as

$$TM = D^\perp \oplus D_\beta$$

- $D^\perp$is anti-invariant, which means that $\phi D^\perp \subset T^\perp M$,
- D_β is a slant, $\beta \neq \frac{\pi}{2}$, implying that the angle between D_β and $\phi(D_\beta)$ remains constant [13].

Remark 1. Let us asume that M is a pseudo slant submanifold of a Bronze Riemannian manifold $(\widetilde{M}, g, \phi)$.

Let p =dim($D^\perp$) and q=dim(D_β). We can distinguish the following six cases:

- When $q = 0$, M is anti-invariant,
- If $p = 0$ and $\beta = 0$, then M is invariant.
- If $p = 0$ and $\beta \in (0, \frac{\pi}{2})$, then M is classified as proper slant.
- When $\beta = \frac{\pi}{2}$, M is anti-invariant.
- If $p \neq 0$ and q≠0 with $\beta = 0$, then M is a semi-invariant.
- If $p \neq 0$ and q≠0 with $\beta \in (0, \frac{\pi}{2})$, then M is consi dered pseudo-slant.

Let μ, represent the orthogonal complement of ϕTM in $T^\perp M$. In this case, $T^\perp M$ can be expressed as the following decomposition:

$$T^\perp M = \phi TM \oplus \mu = ND^\perp \oplus ND_\beta \oplus \mu,$$
$$ND_\beta \perp ND^\perp. \quad (4.5)$$

Theorem 4. Let M be a totally umbilical pseudo-slant submanifold of a locally Bronze Riemannian manifold $(\widetilde{M}, g, \phi)$. D_β is integrable if he following condition hold:

$$g(\nabla_X U, 3TY) - g(\nabla_Y U, 3TX) = g(X, TY)g(K, NU) - g(Y, TX)g(K, NU)$$

for any $X, Y \in \Gamma(D_\theta)$ and $U \in \Gamma(D^\perp)$.

Proof. From (2.1), (3.1), (3.2) and (3.3), we can conclude the following:

$$g(\sigma(X, \phi Y), \phi U) = g\left(A_{\phi U} X, \phi Y\right)$$
$$= g(\nabla_X^\perp \phi U, \phi Y) - g(\tilde{\nabla}_X \phi U, \phi Y)$$
$$= g(\nabla_X^\perp \phi U, \phi Y) - g\left(\left(\tilde{\nabla}_X \phi\right) U, \phi Y\right)$$
$$-g(\phi \tilde{\nabla}_X U, \phi Y)$$
$$= -g(\tilde{\nabla}_X U, \phi^2 Y) = -g(\tilde{\nabla}_X U, (3\phi + I)Y)$$
$$= -g(\nabla_X U, 3\phi Y) - g(\nabla_X U, Y).$$

Thus,

$$g(\nabla_X Y, U) = g(h(X, \phi Y), \phi U) + g(\nabla_X U, 3\phi Y)$$

in the equation above, if we replace X with Y,we obtain the following:

$$g(\nabla_Y X, U) = g(h(Y, \phi X), \phi U) + g(\nabla_Y U, 3\phi X).$$

By subtracting the two equations side by side, we obtain the following:

$$g(\nabla_X Y, U) - g(\nabla_Y X, U)$$
$$= g(h(X, \phi Y), \phi U) + g(\nabla_X U, 3\phi Y)$$
$$-g(h(Y, \phi X), \varphi U) - g(\nabla_Y U, 3\phi X),$$
$$g(U, [X, Y]) = g(\nabla_X U, 3\phi Y) + g(h(X, \phi Y), \phi U)$$
$$-g(\nabla_Y U, 3\phi X) - g(h(Y, \phi X), \phi U).$$

Since D_β is integrable, it follows that:

By using (3,5), we get

$$g(\nabla_X U, 3TY) - g(\nabla_Y U, 3TX) =$$
$$g(g(X, \phi Y)K, \phi U) - g(g(Y, \phi X)K, \phi U) =$$
$$g(X, TY)g(K, NU) - g(Y, TX)g(K, NU).$$

Theorem 5. Let M be a totally umbilical pseudo-slant submanifold of a locally Bronze Riemannian manifold $(\widetilde{M}, g, \phi)$. n is parallel if and only if

$$g(V, NX)U + g(U, NX)V = 0$$

for any $V, U \in \Gamma(T^\perp M), X \in \Gamma(TM)$.

Proof. If n is parallel, then $\nabla n = 0$. From (3.3) and (3.21) , we obtain the following:

$$0 = g(h(tV, X) + NA_V X, U)$$
$$= g(A_U tV, X) + g(A_V X, tU)$$
$$= g(A_U tV + A_V tU, X)$$
$$= g(h(tV, X), U) + g(h(tU, X), V)$$

By using (3,5), we get

$$0 = g(g(tV, X)K, U) + g(g(tU, X)K, V)$$
$$= g(g(V, NX)K, U) + g(g(U, NX)K, V)$$
$$= g(V, NX)g(K, U) + g(U, NX)g(K, V)$$
$$= g(g(V, NX)U + g(U, NX)V, K)$$

for any V, $U \in \Gamma(T^\perp M)$ and for any $X \in \Gamma(TM)$.

Theorem 6. [3] Let M be a pseudo-slant submanifold in a locally Bronze Riemannian manifold $(\widetilde{M}, g, \phi)$. In this case, $D^\perp$ is integrable if and only if

$$A_{ND^\perp} D^\perp = 0.$$

Theorem 7. Let M be a totally umbilical pseudo-slant submanifold in a locally Bronze Riemannian manifold $(\widetilde{M}, g, \phi)$. In this case, $D^\perp$ is integrable if and only if

$$(\nabla_W T)U = (\nabla_U T)W$$

for all $W, U \in \Gamma(D^\perp)$.

Proof. For all $W, U \in \Gamma(D^\perp)$. Using (3.18), we obtain

$$(\nabla_W T)U = A_{NU} W + th(W, U)$$
$$= A_{NU} W + g(W, U)tK \quad (4,6)$$

Replacing W by U in the above equation, we have

$$(\nabla_U T)W = A_{NW} U h + g(U, W)tK \quad (4,7)$$

Then, (4.6), (4.7), from Theorem 6 and (3,5), we arrive at the conclusion.

Finally, let us provide an example previously developed by the authors to support the topic.

Example 2 . To construct a pseudo-slsnt submanifold of a Bronze Riemannian manifold based on the provided parametrization $\chi(\mathrm{u},\mathrm{v})$, we first need to analyze the given mapping and then define the associated Riemannian structure.the mapping is defined as:

$$\chi(\mathrm{u},\mathrm{v}) = (\mathrm{usin}\alpha, -\mathrm{vsin}\alpha, (3-\cos\alpha)\mathrm{u}, (3 + \cos\alpha)\mathrm{v})$$

this mapping $\chi: \mathbb{R}^2 \rightarrow \mathbb{R}^4$ defines a submanifold M in $\mathbb{R}^4$. To ensure M is a submanifold of $\mathbb{R}^4$,we will consider the tangent vectors and the embedding The parametriziation consist of two parameters (u,v). Next, we calculate the tangent vector of M by differentiating χ with respect to each parameter:

$$\mathrm{e}_1 = \frac{\partial\chi}{\partial\mathrm{u}} = (\sin\alpha, 0, 3-\cos\alpha, 0)$$
$$\mathrm{e}_2 = \frac{\partial\chi}{\partial\mathrm{v}} = (0, -\cos\alpha, 0, 3+\cos\alpha)$$

For the Bronze Riemannian structure ϕ of $\mathbb{R}^4$, the coordinat system is given by (x_1, y_1, x_2, y_2).

$$\phi(\frac{\partial}{x_i}) = \frac{\partial}{y_i}, \phi(\frac{\partial}{y_j}) = \frac{\partial}{x_j}, \qquad 1 \leqslant i,j \leqslant 2$$

then we obtain

$$\phi e_1 = (0, sin\alpha, 0, 3-cos\alpha)$$
$$\phi e_2 = (-cos\alpha, 0, 3+cos\alpha, 0)$$
$$g(\phi\mathrm{e}_1, \mathrm{e}_2) = g(\mathrm{e}_1, \phi\mathrm{e}_2) g(\phi\mathrm{e}_1, \phi\mathrm{e}_2)$$
$$= 3g(\phi\mathrm{e}_1, \mathrm{e}_2) + g(\mathrm{e}_1, \mathrm{e}_2)$$

Thus, this structure is observed to be a Bronze structure.

Through direct calculations, we determine that $\mathrm{D}_\beta = \mathrm{Sp}\{\mathrm{e}_1, \mathrm{e}_2\}$ defines a slant distribution with a slant angle of;

$$\cos\beta = \frac{\mathrm{g}(\mathrm{e}_{1,}\varphi\mathrm{e}_2)}{\|\mathrm{e}_{1,}\|\|\varphi\mathrm{e}_2\|} = \frac{8}{\sqrt{10-6\cos\alpha}.\sqrt{10+6\cos\alpha}}$$
$$= \frac{8}{\sqrt{100-36cos\alpha}}$$
$$\beta = \arccos\left(\frac{8}{\sqrt{100-36cos\alpha}}\right).$$
$$-1 \leq cos\alpha \leq 1, 64 \leq 100 - 36cos\alpha \leq 136,$$

Consequently, M is a 2-dinensionel invariant or proper pseudo slant submanifold of $\mathbb{R}^4$ endowed with its standard Bronze Riemannian structure [3].

5. Conclusion

The study highlights the geometric structure of totally umbilical pseudo-slant submanifolds in Bronze Riemannian manifolds, establishing key existence conditions and properties. The provided examples demonstrate the applicability of the theoretical results.

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