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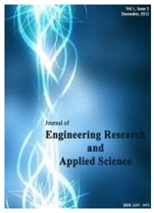

Generic Submanifolds of Para β-Kenmotsu Manifold

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Abstract
The aim of the present paper is to define and study generic submanifolds of para β-Kenmotsu manifold. We investigate the geometry of leaves which arise the definition of a generic submanifold and obtained integrability conditions of distributions. We also consider parallel conditions of projections on generic submanifods of para β-Kenmotsu manifold.

Keywords: *Para β-Kenmotsu manifold, generic submanifolds*

1. Introduction

Complex and contact geometry have many aplication in mathematics and physics. Many geometric properties that ocur in complex structures were examined on contact structures. Moreover, important results were obtained regarding the geometric properties of the contact structures themselves.

Para complex manifold is defined a 2n-dimensional differentiable manifold with endomorphism $J^2 = I$ such that 1-eigen distribution. Similary, a para contact manifold is defined (2n+1)-dimensional diferetiable manifold with $\varphi^2 = I + \eta \otimes \xi$ where φ is (1.1) tensor field, η and ξ ais contact form and characteristic vector field, respectively.

In 1985, Kaneyuki and Williams defined and studied Para-contact manifolds [1]. After Zamkovoy investigated some properties of an almost para-contact metric manifolds and their subclasses [8]. A Para-Kenmotsu manifold is a class of para-contact manifold which were defined by Sinha and Sai Prasad [4] in 1995. After [2], Olszak introduced para β-Kenmotsu manifold.

On the other hand generic submanifolds are generalized semi-invariant submanifolds. A submanifolds of M almost contact manifold $\widetilde{M}$ is called generic submanifold of $\widetilde{M}$ if $H_x = (T_x M) \cap \varphi(T_x M)$ defines a smooth distribution $\forall x \in M$. There many researches on the generic submanifolds as like [3,5, 6,7].

Our aim in the present work to obtain generic submanifolds of para β-Kenmotsu manifold. We define generic submanifolds of para β-Kenmotsu manifold. We are studied integrability and parallel conditions of distributions.

2. Para β-Kenmotsu Manifolds

Let $\overline{M}$ be a *(2n+1)*-dimensional (connected) differentiable manifold endowed with a quadruplet (φ, ξ, η, g), where φ is (1,1)-tensor field, ξ is a vector field, η is a 1-form, and g is a pseudo-Riemannian metric such that

$$\varphi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1 \tag{1}$$

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y)) \tag{2}$$

for all $X, Y \epsilon \Gamma(TM)$. In addition, we have

$$\varphi(\xi) = 0, \;\; \eta o \varphi = 0, \;\; \eta(X) = g(X, \xi). \tag{3}$$

The manifold $\overline{M}$ will be called almost para contact metric, and the quadruplet (φ, ξ, η, g) will be called the almost para contact metric structure on $\overline{M}$. Moreover, a almost para contact metric manifold is normal if $[\varphi, \varphi] - 2d\eta \otimes \xi = 0$ where $[\varphi, \varphi]$ is denoting the Nijenhuis tensor field associated to φ. A normal almost para contact metric manifold is called para contact metric manifold.

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Definition 2.1 Let $\overline{M}$ be an almost para contact metric manifold of dimension (2n+1), with (φ, ξ, η, g). $\overline{M}$ is said to be an almost para β-Kenmotsu manifold if 1-form η are closed and $d\Phi = 2\beta\eta \wedge \Phi$. A normal almost para β-Kenmotsu manifold M is called a para β-Kenmotsu manifold.

If $\overline{M}$ is also normal then we call $\overline{M}$ is called a para β-Kenmotsu manifold. The following theorem gives us the neccesary and sufficient condition for $\overline{M}$ to be para β-Kenmotsu manifold.

Theorem 2.2 Let $(\overline{M}, \varphi, \xi, \eta, g)$ be a para contact metric manifold. $\overline{M}$ is a para β-Kenmotsu manifold if and only if

$$(\overline{\nabla}_X\varphi)Y = \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\} \tag{4}$$

for all $X, Y \in \Gamma(T\overline{M})$.

Corollary 2.3 Let $\overline{M}$ be (2n+1)-dimensional a para β-Kenmotsu manifold with structure (φ, ξ, η, g). Then we have

$$\overline{\nabla}_X\xi = \beta\varphi^2 X \tag{5}$$

for all $\boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{\Gamma(T\overline{M})}$.

3. Submanifolds of Para β-Kenmotsu Manifold

Let $\overline{M}$ be a $(2n+1)-$ dimensional $\beta-$ Kenmotsu manifold. M be a $n-$dimensional submanifold of $\overline{M}$. Then Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y - h(X, Y) \tag{6}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X Y \tag{7}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM)^{\perp}$. h is the second fundamental from of M, $\nabla^{\perp}$ is the connection in the normal bundle and A_V is the Weingarten endomorphism associated with *V*. Shape operator *A* and *t*he second fundamental form h related by

$$g(h(X, Y), V) = g(A_V X, Y) \tag{8}$$

On the other hand, the mean curvature tensor *H* is defined by

$$H = \frac{1}{m}\sum_{k=1}^{m} h(e_k, e_k) \tag{9}$$

where $\{e_1, \ldots, e_m\}$ is a local orthonormal basis of *TM*.

For every tangent vector field X on M we can write

$$\varphi X = PX + QX \tag{10}$$

where PX (resp. QX) denotes the tangential (resp. normal) component of φX. Moreover for every normal vector field N we can state

$$\varphi N = pN + qN \tag{11}$$

where pN in the tangential component of φN and qN is the normal one.

Now, for later use, we establish a result for a submanifold β-Kenmotsu manifold.

Proposition 3.1 Let M be submanifold of β-Kenmotsu manifold $\overline{M}$. Then we have

$$(\nabla_X P)Y = A_{QY}X + ph(X, Y) + \beta g(PX, Y)\xi - \beta\eta(Y)PX \tag{12}$$

$$(\nabla_X Q)Y = qh(X, Y) - h(X, PY) - \beta\eta(Y)QX \tag{13}$$

for all $X, Y \epsilon \Gamma(TM)$

Proposition 3.2 Let M be submanifold of β-Kenmotsu manifold $\overline{M}$. Then we have

$$g(PX, Y) = g(X, PY)$$
$$g(pX, V) = g(X, pV)$$
$$g(qU, V) = g(U, qV)$$

for all $X, Y \in \Gamma(TM), U, V \in \Gamma(TM^{\perp})$.

Proposition 3.3 Let M be submanifold of β-Kenmotsu manifold $\overline{M}$. Then we have

$$P\xi = 0, \qquad Q\xi = 0$$
$$QT + qN = 0$$
$$pn + Pt = 0.$$

4. Generic Submanifolds of Para β-Kenmotsu Manifold

Defination 4.1 Let M be submanifold of β-Kenmotsu manifold $\overline{M}$ with characteristic vector ξ is tangent to M. If the maximal invariant subspace under φ, orthogonal to ξ to $T_x(M)$.

$$H_x = T_x(M) \cap \varphi T_x(M), \quad x \in M$$

defines a differentiable distribution of $T_x(M)$, then M is called generic submanifold of $\overline{M}$.

For a generic submanifold M in a β-Kenmotsu manifold $\overline{M}$, the orthogonal complementary distribution ${H_x}^{\perp}$ called the purely real distribution, if it satisfies

$$H_x \perp {H_x}^{\perp}, \; TH_x \subset {H_x}^{\perp}$$
$$H_x \cap \varphi {H_x}^{\perp} = 0.$$

Let μ_x be the vector space of holomorphic normal vectors to M at x, or simply the holomorphic normal space of M at x, i.e.,

$$\mu_x = T_x{}^{\perp} \cap \varphi T_x{}^{\perp}.$$

Then μ_x defines a differentiable vector subbundle of $T_x{}^{\perp}M$. It is easy to verify that

$$TM^{\perp} = P(H^{\perp}) \oplus \mu, \qquad pTM^{\perp} = H^{\perp}$$

and

$$g(P(H^{\perp}), \mu) = 0.$$

A vector subbundle of μ of the normal bundle $TM^{\perp}$ is said to be parallel (in the normal bundle) if

$$\nabla_{\mathrm{x}}{}^{\perp}\mathrm{K} \in \mu$$

for any $X \in \Gamma(TM)$ and any local cross-section K in μ.

5. Geometry of Distributions

Lemma 5.1 M be submanifold of β-Kenmotsu manifold $\overline{\mathrm{M}}$. Then

$$g(\varphi h(X,U), K) = g(h(\varphi X, U), K)$$

for all $X \in \Gamma(H), U \in \Gamma(TM)$ *and* $K \in \Gamma(\mu)$.

Theorem 5.2 M be submanifold of β-Kenmotsu manifold $\overline{\mathrm{M}}$. Then the distribution H is always integrable.

Proof. For all $X, Y \in \Gamma(H)$, we have

$$g([X,Y], \xi) = g(\nabla_{\mathrm{X}}Y - \nabla_{\mathrm{Y}}X, \xi)$$
$$= -g(Y, \nabla_{\mathrm{X}}\xi) + g(X, \nabla_{\mathrm{Y}}\xi).$$

Then last equation using (5), we get

$$g([X,Y], \xi) = 0$$

which our assertion.

Theorem 5.3 M be submanifold of β-Kenmotsu manifold $\overline{\mathrm{M}}$. Then the distribution $H^{\perp} \oplus \{\xi\}$ is integrable if and only if

$$A_{PZ}W - A_{PW}Z + \nabla_{\mathrm{Z}}\mathrm{PW} - \nabla_{\mathrm{W}}\mathrm{PZ} \in H^{\perp}$$

for all $Z, W \in \Gamma(H^{\perp})$.

Proof. For all $U, V \in \Gamma(H^{\perp})$, $X \in \Gamma(H)$, using equation (6), we get

$$g([U,V], \varphi X) = g(\overline{\nabla}_{\mathrm{U}}V, \varphi X) - g(\overline{\nabla}_{\mathrm{V}}U, \varphi X).$$

On the other hand, some elementary calculations,

$$g([U,V], \varphi X) = g(\overline{\nabla}_{\mathrm{U}}\varphi V, X) - g(\overline{\nabla}_{\mathrm{V}}\varphi U, X).$$

Then from (10)

$$g([U,V], \varphi X) = g(\overline{\nabla}_{\mathrm{U}}PV + \overline{\nabla}_{\mathrm{U}}QV, X) + g(\overline{\nabla}_{\mathrm{V}}PU + \overline{\nabla}_{\mathrm{V}}QU, X).$$

Finally, using (6) and (7) we have

$$g([U,V], \varphi X) = g(A_{PZ}W, X) - g(A_{PW}Z, X)$$
$$+ g(\nabla_{\mathrm{Z}}\mathrm{PW}, \mathrm{X}) - g(\nabla_{\mathrm{W}}\mathrm{PZ}, \mathrm{X})$$

which gives proof.

Theorem 5.4 M be submanifold of β-Kenmotsu manifold $\overline{\mathrm{M}}$. Then the distribution $H \oplus \{\xi\}$ and its leaves are totally geodesic in M if and only if

$$g(h(H,H), NH^{\perp}) = 0.$$

Proof. For all $X, Y \in \Gamma(H)$, $U \in \Gamma(H^{\perp})$, from (8) and (6), we have

$$g(h(X,Y), QU) = -g(\overline{\nabla}_{\mathrm{X}}\mathrm{QU}, Y).$$

Then using (10) and after some elementary calculations, we get

$$g(h(X,Y), QU) = g(\overline{\nabla}_{\mathrm{X}}\mathrm{U}, \varphi Y).$$

Since $\nabla_{\mathrm{H}}H^{\perp} \subset H^{\perp}$ we have

$$g(\overline{\nabla}_{\mathrm{X}}\mathrm{U}, \varphi Y) = 0$$

which proves assertion.

Lemma 5.6 M be submanifold of β-Kenmotsu manifold $\overline{\mathrm{M}}$. Then T is parallel if and only if

$$A_{QX}Y = A_{QY}X$$

for all $X, Y \in \Gamma(TM)$.

Proposition 5.7 M be submanifold of β-Kenmotsu manifold $\overline{\mathrm{M}}$. Then N is parallel if and only if

$$A_{qV}X = A_V PX$$

for all $X \in \Gamma(TM), K \in \Gamma(TM^{\perp})$.

Proof. For all $X, Y \in \Gamma(TM), K \in \Gamma(TM^{\perp})$ using (13), we have

$$g\big((\nabla_X Q)Y, K\big) = g(qh(X,Y) - h(X,PY)$$
$$- \eta(Y)QX, K)$$
$$= g(h(X,Y), qK) - g(h(X,PY), K).$$

Then from (8) we get,

$$g\big((\nabla_X Q)Y, K\big) = g\big(A_{qK}X, Y\big) - g(A_K PX, Y)$$

which gives our assertion.

Proposition 5.8 M be submanifold of β-Kenmotsu manifold $\overline{\mathrm{M}}$. Then n is parallel if and only if

$$A_V PU = A_U pV$$

for all $U \in \Gamma(TM), V \in \Gamma(TM^{\perp})$.

Proof. For all $X, Y \in \Gamma(TM), K \in \Gamma(TM^{\perp})$, we have

$$g\big((\nabla_X q)K, Y\big) = g(QA_K X - h(X, pK), Y).$$

Using (8) and (10) we get

$$g\big((\nabla_X q)K, Y\big) = g(A_K X, PY) - g(A_Y PK, X)$$
$$= -g(X, A_K PY) + g(PK, A_Y X)$$

which conclude proof.

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